

ON THE SELF-SHRINKING SYSTEMS IN ARBITRARY CODIMENSIONAL SPACES

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ABSTRACT. In this paper, we discuss the self-shrinking systems in higher codimensional spaces. We mainly obtain several Bernstein type results and a sharp growth estimate.
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1. INTRODUCTION

Let M be an n -dimensional manifold immersed into the Euclidean space \mathbb{R}^{n+m} . Denote the immersed map by $X : M \rightarrow \mathbb{R}^{n+m}$. Let $(\cdots)^N$ be the projective map from the trivial bundle $\mathbb{R}^{n+m} \times M$ onto the normal bundle over M . Then we can define the mean curvature vector of the immersed manifold M into \mathbb{R}^{n+m} , seeing [20],

$$H = \sum_{i=1}^n \bar{\nabla}_{e_i}^N e_i.$$

Here, $\{e_i\}_{i=1}^n$ is a local orthonormal tangent frame of M , $\bar{\nabla}$ is the canonical connection of \mathbb{R}^{n+m} and $\bar{\nabla}^N$ is the connection of normal bundle NM . If we let the position vector X move in the direction of the mean curvature vector H , then it gives the *mean curvature flow*, namely,

$$(1.1) \quad \frac{\partial X}{\partial t} = H, \quad \text{on } M \times [0, T).$$

Here, $[0, T)$ is the maximal finite time interval on which the flow exists.

The immersed manifold M is said to be a *self-shrinker* (see [11] or [15] for details), if it satisfies a quasi-linear elliptic system,

$$(1.2) \quad H = -X^N.$$

Self-shrinkers are an important class of solutions to (1.1). Not only are they shrinking homothetic under mean curvature flow (see [5] for detail), but also they describe all possible blow ups at a given singularity of a mean curvature flow.

Now, we give a roughly brief review about the self-shrinkers. For curves case, U. Abresch and J. Langer [1] gave a complete classification of all solutions to (1.2). These curves are now called Abresch-Langer curves.

For codimension 1 case, K. Ecker and G. Huisken [7] showed that a self-shrinker is a hyperplane, if it is an entire graph with polynomial volume growth. Let \vec{n} be the unit outward normal vector of M^n in \mathbb{R}^{n+1} and $|B|$ be the norm of the second fundamental form of M . In [11] and [12], G. Huisken gave the classification theorem that the only possible smooth self-shrinkers in \mathbb{R}^{n+1} satisfying mean curvature of M , $\operatorname{div}(\vec{n}) \geq 0$, $|B|$ bounded and polynomial volume growth are isometric to $\Gamma \times \mathbb{R}^{n-1}$ or $\mathbb{S}^k \times \mathbb{R}^{n-k}$ ($0 \leq k \leq n$). Here, Γ is an Abresch-Langer curve and \mathbb{S}^k is a k -dimensional sphere. In general, the classification of self-shrinkers seems difficult. However T.H. Colding and W.P. Minicozzi

II [4] offer a possibility. Recently, They [5], [6] showed that a long-standing conjecture of Huisken is right. The conjecture is classifying the singularities of mean curvature flow starting at a generic closed embedded surface in \mathbb{R}^3 . They also proved that G. Huisken's classification theorem still holds without the assumption that $|B|$ is bounded. Meanwhile, L. Wang [17] proved that an entirely graphic self-shrinker should be a hyperplane without any other assumption.

Another special case is that the self-shrinker is a Lagrangian graph. In [2], A. Chau, J.Y. Chen, and W.Y. He firstly study Lagrangian self-shrinkers in Euclidean space. Recently, R. Huang and Z. Wang [9] obtained the following two results for Lagrangian graphs. In pseudo-Euclidean space, if the Hessian of the potential function has order two decay in the infinity, the self-shrinker defined by the potential is a linear subspace. In Euclidean space, if the potential function is convex or concave, the corresponding self-shrinker is a linear subspace. After that, A. Chau, J.Y. Chen and Y. Yuan [3] improved and generalized previous results. In Euclidean space, they drop the assumption that the potential should be convex or concave. In pseudo-Euclidean, using different method, they proved a similar result. They also generalized their pseudo-Euclidean result to complex case, which relates to a special class of self-shrinking Kähler-Ricci solitons.

In arbitrary codimension case, K. Smozyk in [15] proved two results. Suppose the manifold M^m is a compact self-shrinker, then M is spherical if and only if $H \neq 0$ and its principal normal vector $\nu = H/|H|$ is parallel in the normal bundle. Suppose M^m is a complete connected self-shrinker with $H \neq 0$, parallel principal normal and having uniformly bounded geometry, then M must be $\Gamma \times \mathbb{R}^{m-1}$ or $\tilde{M}^r \times \mathbb{R}^{m-r}$. Here Γ is an Abresch-Langer curve and \tilde{M} is a minimal submanifold in sphere.

The study of higher codimensional self-shrinkers seems difficult for us. Hence, in the present paper, we only consider the simplest case that M is a smooth graph.

Assume M to be a codimension m smooth graph in \mathbb{R}^{n+m} ,

$$M = \{(x_1, \dots, x_n, u^1, \dots, u^m) \in \mathbb{R}^{n+m}; x_i \in \mathbb{R}, u^\alpha = u^\alpha(x_1, \dots, x_n)\},$$

where $i = 1, \dots, n$ and $\alpha = 1, \dots, m$. Denote

$$(1.3) \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n; \quad u = (u^1, \dots, u^m) \in \mathbb{R}^m.$$

In Euclidean and pseudo-Euclidean space, denote the metric on M be $g = \sum_{i,j=1}^n g_{ij} dx_i dx_j$, and

$$g_{ij} = \delta_{ij} \pm \sum_{\alpha=1}^m u_i^\alpha u_j^\alpha.$$

Here, $u_i^\alpha = \frac{\partial u^\alpha}{\partial x_i}$, "+" is chosen in Euclidean space and "-" is chosen by space-like submanifolds in pseudo-Euclidean space with index m , which means the background metric in \mathbb{R}^{n+m} is

$$ds^2 = \sum_{i=1}^n dx_i^2 - \sum_{\alpha=1}^m dx_{n+\alpha}^2,$$

seeing [20] for details. Then, (1.2) is equivalent to the following elliptic system

$$(1.4) \quad \sum_{i,j=1}^n g^{ij} u_{ij}^\alpha = -u^\alpha + x \cdot Du^\alpha.$$

Here, $Du^\alpha = (u_1^\alpha, \dots, u_n^\alpha)$, and \cdot denotes the canonical inner product in \mathbb{R}^n . The details of the calculation will appear in section 2.

In section 3, a special class of functions is important. It is,

Definition 1. Assume ϕ to be a C^2 -function defined on \mathbb{R}^n . We call that ϕ is a strong-sub-harmonic (shortly, SSH) function, if ϕ satisfies

$$(1.5) \quad \sum_{i,j=1}^n g^{ij} \phi_{ij} - x \cdot D\phi \geq \varepsilon \sum_{i,j=1}^n g^{ij} \phi_i \phi_j,$$

for some small positive constant ε .

Based on the Lemma 5, we know that the only possible SSH functions are constants in \mathbb{R}^n . Hence, our crucial problem becomes to find appropriate SSH-functions. We prove that the volume element function is SSH, if one of the following three assumptions is satisfied. The first one is that the multiple of two different eigenvalues of the singular value of du (c.f. [18]) is no more than one. The second one is that the volume element function is less than some positive constant $\beta < 9$. The last one is a communication formula which includes the normal bundle flat case and codimension 1 case. Then, the volume element function is a constant which implies Bernstein type results. The corresponding minimal submanifolds cases have appeared in [13], [16] and [18]. At the end of this section, we prove that there is no non-trivial rotationary symmetric self-shrinking graph, but for submanifolds, we have non-trivial examples. S. Kleene and N.M. Møller [14] gave the classification for complete embedded revolutionary hypersurfaces.

In section 4, we give the sharp growth estimate of u^α . Obviously, the linear functions satisfy (1.4). In Euclidean space, we have the following estimate.

Theorem 2. The solution u^α of system (1.4) defining a graph in Euclidean space is linear growth. In fact, we have the following estimate,

$$(1.6) \quad |u(x)|^2 \leq \left(\frac{2|x|^2}{3n} + 1\right) \left(\sup_{|x| \leq 2\sqrt{3n}} |u(x)|^2 + 12n\right),$$

where $|u(x)|^2 = \sum_{\alpha=1}^m (u^\alpha(x))^2$.

Then, we generalize the above estimate to the following linear elliptic system,

$$(1.7) \quad \sum_{i,j=1}^n a^{ij} u_{ij}^\alpha = -u^\alpha + x \cdot Du^\alpha,$$

with two extra assumptions. One is that the coefficient matrix (a^{ij}) have a uniform upper bound. Namely, there is some positive constant σ , such that

$$(1.8) \quad \sum_{i,j=1}^n a^{ij} \xi_i \xi_j \leq \sigma |\xi|^2$$

holds for any vector $\xi \in \mathbb{R}^n$. The other is that there be three positive constants r_0 , c and τ , such that for $|x| \geq r_0$, we have

$$(1.9) \quad \sum_{i,j=1}^n \sum_{\alpha=1}^m a^{ij}(x) u_i^\alpha(x) u_j^\alpha(x) \leq c |x|^{2\tau-2}.$$

Roughly speaking, we prove that any solution u^α satisfying (1.7), (1.8) and (1.9) has polynomial growth. See Theorem 18 for details.

In the last part, we discuss the Bernstein type theorems for the space-like self-shrinking graph in pseudo-Euclidean space with index m . Firstly, We prove that, if the metric has a positive lower bound, the graph should be a linear subspace. If a self-shrinking graph also pass through the origin, we have the following better result.

Theorem 3. *Assume u^α to be a solution of the system (1.4), defining a graph in pseudo-Euclidean space with index m . Further, assume that $u^\alpha(0) = 0$ and that $\det(g)$ has subexponential decay in the sense that*

$$\lim_{|x| \rightarrow \infty} \frac{\log \det(g_{ij}(x))}{|x|} = 0.$$

Then M is a linear subspace.

This paper is organized as follows. In the second section, we calculate explicitly the self-shrinking system in higher codimensional spaces. In the third section, we derive several Bernstein type results in Euclidean space. In the forth section, we obtain the sharp growth estimate for self-shrinking function u^α in Euclidean space. Then, we generalize it to some class of linear elliptic systems. In the last section, we obtain two results about space-like self-shrinking graph in pseudo-Euclidean space with index m .

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2. THE SELF-SHRINKING SYSTEMS

From now on, Einstein convention of summation over repeated indices will be adopted. We also assume that Latin indices i, j, \dots , Greek indices α, β, \dots and capital Latin indices A, B, \dots take values in the sets $\{1, \dots, m\}$, $\{m+1, \dots, m+n\}$ and $\{1, \dots, m+n\}$ respectively.

Firstly, we calculate the system of the self-shrinkers in higher codimensional spaces. For the graph M , denote

$$X = (x_1, \dots, x_n, u^1, \dots, u^m) = (x, u).$$

Let $\{E_A\}_{A=1}^{n+m}$ be the canonical orthonormal basis of \mathbb{R}^{n+m} . Namely, every component of the vector E_A is 0, except that the A -th component is 1. Then

$$e_i = E_i + \sum_{\alpha} u_i^\alpha E_{n+\alpha}, \quad \text{for } i \in \{1, \dots, n\}$$

give a tangent frame on M .

In Euclidean space, the metric on M is

$$g_{ij} = \langle e_i, e_j \rangle = \delta_{ij} + \sum_{\alpha} u_i^\alpha u_j^\alpha.$$

Here, $\langle \cdot, \cdot \rangle$ is the canonical inner product in \mathbb{R}^{n+m} . Then there are m linear independent unit normal vectors,

$$n_\alpha = \frac{1}{(1 + |Du^\alpha|^2)^{1/2}} \left(- \sum_i u_i^\alpha E_i + E_{n+\alpha} \right), \quad \text{for } \alpha \in \{1, \dots, m\}.$$

Note that $\{n_\alpha\}$ are not necessarily orthogonal with each other. We always denote $Du^\alpha = \sum_i u_i^\alpha E_i$.

Now we define the α -th mean curvature component is

$$(2.1) \quad H^\alpha = \langle H, n_\alpha \rangle.$$

Then, we have

$$\begin{aligned} \langle H, n_\alpha \rangle &= g^{ij} \langle \bar{\nabla}_{e_i} e_j, n_\alpha \rangle = g^{ij} \langle u_{ij}^\beta E_{n+\beta}, n_\alpha \rangle \\ &= \frac{g^{ij} u_{ij}^\alpha}{(1 + |Du^\alpha|^2)^{1/2}}. \end{aligned}$$

Then by (1.2),

$$(2.2) \quad \frac{g^{ij} u_{ij}^\alpha}{(1 + |Du^\alpha|^2)^{1/2}} = H^\alpha = -\langle X, n_\alpha \rangle = \frac{-u^\alpha + x \cdot Du^\alpha}{(1 + |Du^\alpha|^2)^{1/2}}.$$

We obtain (1.4).

In pseudo-Euclidean space with index m , for the space-like graphes, the metric and the normal directions are

$$(2.3) \quad g_{ij} = \delta_{ij} - \sum_\alpha u_i^\alpha u_j^\alpha, \quad \text{and} \quad n_\alpha = \frac{1}{(1 - |Du^\alpha|^2)^{1/2}} (\sum_\alpha u_i^\alpha E_i + E_{n+\alpha}).$$

Then we obtain a similar system as (1.4), where we replace the metric g_{ij} by (2.3).

Remark 4. For a Lagrangian graph, $m = n$, and there is some potential function v (c.f. [8]) , such that

$$u^\alpha = \frac{\partial v}{\partial x_\alpha}.$$

Then inserting it into (1.4), and integrating, we obtain the equations of Lagrangian self-shrinkers in Euclidean and pseudo-Euclidean space. They are

$$(2.4) \quad \text{tr} \arctan D^2 v = -2v + x \cdot Dv,$$

or

$$(2.5) \quad \frac{1}{2} \text{tr} \ln \frac{I + D^2 v}{I - D^2 v} = -2v + x \cdot Dv,$$

respectively. Here, tr means taking the trace of matrices.

For the first equation, we let

$$w(y) = 2v\left(\frac{1}{\sqrt{2}}y\right).$$

Then (2.4) becomes the equation (2) in [2]. For the second equation, first, we let

$$\eta(y) = \frac{4}{n} v\left(\frac{\sqrt{n}}{2}y\right).$$

The equation becomes

$$\text{tr} \ln \frac{I + D^2 \eta}{I - D^2 \eta} = n(-\eta + \frac{1}{2}y \cdot D\eta).$$

Then, using Lewy rotation [21],

$$\begin{cases} \bar{x} = \frac{x - D\eta(x)}{\sqrt{2}} \\ Dw(\bar{x}) = \frac{x + D\eta(x)}{\sqrt{2}} \end{cases},$$

we get

$$\ln \det D^2 w = \operatorname{tr} \ln D^2 w = n(-w + \frac{1}{2} \bar{x} \cdot Dw).$$

In the last equality, we use a similar trick appearing in [9]. Then we obtain the self-shrinking equations of Lagrangian mean curvature flow in pseudo-Euclidean space. This equation appears in [9] and [3] at first.

3. SOME BERNSTEIN TYPE RESULTS IN EUCLIDEAN SPACE

For any function ϕ on \mathbb{R}^n , denote an operator

$$(3.1) \quad L_a \phi = a^{ij} \phi_{ij} - x \cdot D\phi.$$

Here, (a^{ij}) is the inverse of a positive definite matrix (a_{ij}) at every point in \mathbb{R}^n . The critical point to obtain Bernstein type results is the following lemma. It seems that the second inequality of the following lemma similar to some stability condition. The proof of the lemma slightly modifies from the last part of [9]. But for the readers' convenient, we include it here.

Lemma 5. *Let the minimum eigenvalue of the matrix (a_{ij}) be $\nu(x)$. Assume*

$$(3.2) \quad \liminf_{|x| \rightarrow +\infty} \nu(x) |x|^2 > n.$$

For any smooth function ϕ , if there is a small positive constant ε , such that,

$$L_a \phi \geq \varepsilon a^{ij} \phi_i \phi_j,$$

then ϕ is a constant.

Proof. For $0 < k < 1$, let

$$\eta(x) = \begin{cases} 1 & |x| \leq R_0 \\ -k(|x|^2 - R_0^2) + 1 & |x| \geq R_0 \end{cases}.$$

Here R_0 is a constant which will be determined later. Then the function $\eta e^{C\phi}$ achieves its maximum in the bounded set

$$\{x \in \mathbb{R}^n; \eta > 0\},$$

where C is a positive constant which also will be determined later. If the maximum point p is in the set $\{x \in \mathbb{R}^n; |x| > R_0\}$, then at p , we have

$$(3.3) \quad \eta_i + C\eta\phi_i = 0.$$

At p , using (3.3), we have,

$$\begin{aligned}
 (3.4) \quad e^{-C\phi} a^{ij} (\eta e^{C\phi})_{ij} &= a^{ij} \eta_{ij} + 2C a^{ij} \eta_i \phi_j + C \eta a^{ij} \phi_{ij} + C^2 \eta a^{ij} \phi_i \phi_j \\
 &\geq -2k \sum_i a^{ii} - C^2 \eta a^{ij} \phi_i \phi_j + C \eta (x \cdot D\phi + \varepsilon a^{ij} \phi_i \phi_j) \\
 &= -x \cdot D\eta - 2k \sum_i a^{ii} + (C\varepsilon - C^2) \eta a^{ij} \phi_i \phi_j \\
 &= 2k(|x|^2 - \sum_i a^{ii}) + (C\varepsilon - C^2) \eta a^{ij} \phi_i \phi_j.
 \end{aligned}$$

By condition (3.2), there is a sufficient large radius R_1 , such that in $\mathbb{R}^n \setminus B_{R_1}$, we have

$$a^{ii} < \frac{|x|^2}{n},$$

for any i , where B_{R_1} denotes an open ball of radius R_1 centred at origin. Using (3.4) and taking the constant C sufficient small, we obtain a contradiction if $p \in (\mathbb{R}^n \setminus B_{R_1}) \cap \{x \in \mathbb{R}^n; |x| > R_0\}$.

Assume that the function ϕ is not a constant in \mathbb{R}^n . Then there is a ball B_{R_0} with radius $R_0 \geq R_1$, such that the function ϕ is not a constant in B_{R_0} . Suppose that ϕ achieves its maximum value in B_{R_0} . Since $L_a \phi \geq 0$, applying strong maximum principle, we obtain ϕ is a constant, which is a contradiction. Hence, ϕ achieves its maximum value only on the boundary ∂B_{R_0} . Similarly, in $B_{\sqrt{R_0^2+1}}$, ϕ also achieves its maximum value only on the boundary $\partial B_{\sqrt{R_0^2+1}}$. We assume that the points p_1 and p_2 are maximum value points with respect to ∂B_{R_0} and $\partial B_{\sqrt{R_0^2+1}}$, namely,

$$\max_{\overline{B}_{R_0}} \phi = \phi(p_1), \quad \max_{\overline{B}_{\sqrt{R_0^2+1}}} \phi = \phi(p_2).$$

Then

$$\phi(p_1) \leq \phi(p_2).$$

But the equality is not valid. In fact, if the equality holds, then the function ϕ achieves its maximum value in the interior of the domain $B_{\sqrt{R_0^2+1}}$, which is a contradiction. Thus, we can choose k sufficiently small, such that

$$(\eta e^{C\phi})(p_1) = (e^{C\phi})(p_1) < ((1-k)e^{C\phi})(p_2) = (\eta e^{C\phi})(p_2).$$

This means that, for fixed ϕ , we can choose suitable k , such that the maximum value of $\eta e^{C\phi}$ only occurs in the set

$$\{x \in \mathbb{R}^n \mid |x| > R_0 \geq R_1\}.$$

But we have proved that it is impossible. Thus, all above discussions imply the function ϕ should be a constant. \square

Remark 6. If the matrix (a_{ij}) is the induced metric of a graph in Euclidean space, we can drop the condition (3.2). In fact, Lemma 5 says that any SSH-function should be a constant in Euclidean space.

In [3], A. Chau, J.Y. Chen and Y. Yuan proved the following theorem. Here, we give another proof.

Theorem 7. For a Lagrangian graph, every self-shrinker should be a linear subspace.

Proof. By [3], the phrase function Θ of a Lagrangian graph satisfies

$$g^{ij}\Theta_{ij} - x \cdot D\Theta = 0.$$

Then

$$g^{ij}(e^\Theta)_{ij} - x \cdot D(e^\Theta) = e^{-\Theta} g^{ij}(e^\Theta)_i (e^\Theta)_j.$$

Since Θ is bounded, e^Θ is SSH. By Lemma 5, Θ is a constant. Then, by the same argument of [3], we obtain the result. \square

A simple result is the following.

Proposition 8. *If f is a nonnegative function satisfying*

$$(3.5) \quad g^{ij}f_{ij} = -f + x \cdot Df,$$

then $f \equiv 0$.

Proof. Since $f \geq 0$, we consider the function

$$\phi = e^{-f}.$$

Then, we have

$$\begin{aligned} g^{ij}\phi_{ij} &= e^{-f}g^{ij}f_i f_j - e^{-f}g^{ij}f_{ij} \\ &= e^{-f}g^{ij}f_i f_j + e^{-f}f - e^{-f}x \cdot Df. \end{aligned}$$

It is

$$\begin{aligned} g^{ij}\phi_{ij} - x \cdot D\phi &= e^{-f}g^{ij}f_i f_j + e^{-f}f \\ &\geq e^{-2f}e^f g^{ij}f_i f_j \\ &\geq g^{ij}\phi_i \phi_j. \end{aligned}$$

So ϕ is SSH. By Lemma 5, we obtain that e^{-f} is a constant. Namely, f is a constant. But (3.5) tells us that the constant must be 0. \square

Corollary 9. *For the system (1.4), assume that every u^α is only in one side of some hyperplane. Namely, there are constant vectors b^α , such that*

$$u^\alpha \geq b^\alpha \cdot x, \quad \text{for } \alpha = 1, \dots, m.$$

Then M is a linear subspace.

Proof. In Proposition 8, let $f = u^\alpha - b^\alpha \cdot x$. Then $f = 0$, this means that u^α is a linear function and M is a linear subspace. \square

A possible SSH-function is the volume element function. See [18] and [21]. In Euclidean space, we let

$$(3.6) \quad \phi = \ln \det(g_{ij}).$$

Then

$$\begin{aligned} (3.7) \quad g^{ij}\phi_{ij} &= -2 \sum_{\alpha, \beta} g^{ij}g^{pk}g^{ql}u_{pi}^\alpha u_q^\alpha u_{kj}^\beta u_l^\beta - 2 \sum_{\alpha, \beta} g^{ij}g^{pk}g^{ql}u_{qi}^\alpha u_p^\alpha u_{kj}^\beta u_l^\beta \\ &\quad + 2 \sum_{\alpha} g^{ij}g^{pq}u_{pi}^\alpha u_{qj}^\alpha + 2 \sum_{\alpha} g^{ij}g^{pq}u_{pij}^\alpha u_q^\alpha. \end{aligned}$$

By system (1.4), we have

$$\begin{aligned} g^{ij}u_{ijp}^\alpha &= x \cdot Du_p^\alpha + g^{ik}g^{jl}g_{klp}u_{ij}^\alpha \\ &= x \cdot Du_p^\alpha + 2 \sum_{\beta} g^{ik}g^{jl}u_{kp}^\beta u_l^\beta u_{ij}^\alpha. \end{aligned}$$

Then by (3.7), we get

$$\begin{aligned} g^{ij}\phi_{ij} &= -2 \sum_{\alpha, \beta} g^{ij}g^{pk}g^{ql}u_{pi}^\alpha u_q^\alpha u_{kj}^\beta u_l^\beta + 2 \sum_{\alpha, \beta} g^{ij}g^{pk}g^{ql}u_{qi}^\alpha u_p^\alpha u_{kj}^\beta u_l^\beta \\ &\quad + 2 \sum_{\alpha} g^{ij}g^{pq}u_{pi}^\alpha u_{qj}^\alpha + 2 \sum_{\alpha} g^{pq}u_q^\alpha x \cdot Du_p^\alpha. \end{aligned}$$

It is

$$\begin{aligned} L_g(\phi) &= -2 \sum_{\alpha, \beta} g^{ij}g^{pk}g^{ql}u_{pi}^\alpha u_q^\alpha u_{kj}^\beta u_l^\beta + 2 \sum_{\alpha, \beta} g^{ij}g^{pk}g^{ql}u_{qi}^\alpha u_p^\alpha u_{kj}^\beta u_l^\beta \\ &\quad + 2 \sum_{\alpha} g^{ij}g^{pq}u_{pi}^\alpha u_{qj}^\alpha. \end{aligned}$$

By simply calculation, the operator L_g is invariant under the orthonormal transformations from \mathbb{R}^n to \mathbb{R}^n and from \mathbb{R}^m to \mathbb{R}^m . Then, at any fixed point, we can choose a coordinate system $\{x_1, \dots, x_n\}$ on \mathbb{R}^n and $\{u^1, \dots, u^m\}$ on \mathbb{R}^m , such that

$$(3.8) \quad \frac{\partial u^\alpha}{\partial x_i} = \lambda_i \delta_i^\alpha,$$

where $\lambda_i \geq 0$ are the singular values of du , and $\lambda_i = 0$ for $i \in \{\min\{m, n\}+1, \dots, \max\{m, n\}\}$. If $n > m$, we let $u_{ij}^A = 0$ for convenience, where $A \in \{m+1, \dots, n\}$. Then, we get

$$\begin{aligned} (3.9) \quad L_g(\phi) &= -2 \sum_{i,p,q} \frac{\lambda_q^2 (u_{pi}^q)^2}{(1+\lambda_i^2)(1+\lambda_p^2)(1+\lambda_q^2)} + 2 \sum_{\alpha, i, p} \frac{(u_{pi}^\alpha)^2}{(1+\lambda_i^2)(1+\lambda_p^2)} \\ &\quad + 2 \sum_{i,p,q} \frac{\lambda_p \lambda_q u_{qi}^p u_{pi}^q}{(1+\lambda_i^2)(1+\lambda_p^2)(1+\lambda_q^2)}. \end{aligned}$$

Now we have the following Theorem.

Theorem 10. *For the system (1.4), if one of the following three assumptions holds:*

- (i) $\lambda_i \lambda_j \leq 1$ for any $i \neq j$;
 - (ii) $\det(g_{ij}) \leq \beta < 9$, where β is a positive constant;
 - (iii) $u_{qi}^\alpha g^{ij} u_{jp}^\beta = u_{qi}^\beta g^{ij} u_{jp}^\alpha$ for any α, β ;
- then M is a linear subspace.

Proof. (i) Since

$$2\lambda_p \lambda_q u_{qi}^p u_{pi}^q \leq 2|u_{qi}^p u_{pi}^q| \leq (u_{qi}^p)^2 + (u_{pi}^q)^2,$$

for $p \neq q$, then by (3.9), we get

$$\begin{aligned}
(3.10) \quad L_g \phi &\geq -2 \sum_{i,p,q} \frac{\lambda_q^2 (u_{pi}^q)^2}{(1+\lambda_i^2)(1+\lambda_p^2)(1+\lambda_q^2)} + 2 \sum_{i,p,q} \frac{(u_{pi}^q)^2 (1+\lambda_q^2)}{(1+\lambda_i^2)(1+\lambda_p^2)(1+\lambda_q^2)} \\
&\quad -2 \sum_{i,p \neq q} \frac{(u_{pi}^q)^2}{(1+\lambda_i^2)(1+\lambda_p^2)(1+\lambda_q^2)} + 2 \sum_{i,p} \frac{\lambda_p^2 (u_{pi}^p)^2}{(1+\lambda_i^2)(1+\lambda_p^2)^2} \\
&= 2 \sum_{i,p} \frac{(u_{pi}^p)^2}{(1+\lambda_i^2)(1+\lambda_p^2)^2} + 2 \sum_{i,p} \frac{\lambda_p^2 (u_{pi}^p)^2}{(1+\lambda_i^2)(1+\lambda_p^2)^2} \\
&\geq \sum_i \frac{2}{n(1+\lambda_i^2)} \left(\sum_p \frac{\lambda_p u_{pi}^p}{1+\lambda_p^2} \right)^2 = \frac{1}{2n} g^{ij} \phi_i \phi_j,
\end{aligned}$$

where we have used the Cauchy inequality in the second inequality. By Lemma 5, ϕ is a constant. Then the above inequality implies $u_{pi}^\alpha = 0$. Then u^α are linear functions.

(ii) By (3.9), we have

$$\begin{aligned}
L_g(e^{\phi/2}) &= e^{\phi/2} \frac{1}{2} L_g(\phi) + e^{\phi/2} \frac{1}{4} g^{ij} \phi_i \phi_j \\
&= -e^{\phi/2} \sum_{i,p,q} \frac{\lambda_q^2 (u_{pi}^q)^2}{(1+\lambda_i^2)(1+\lambda_p^2)(1+\lambda_q^2)} + e^{\phi/2} \sum_{\alpha,i,p} \frac{(u_{pi}^\alpha)^2}{(1+\lambda_i^2)(1+\lambda_p^2)} \\
&\quad + e^{\phi/2} \sum_{i,p,q} \frac{\lambda_p \lambda_q u_{qi}^p u_{pi}^q}{(1+\lambda_i^2)(1+\lambda_p^2)(1+\lambda_q^2)} + e^{\phi/2} \sum_{i,p,q} \frac{\lambda_p \lambda_q u_{pi}^p u_{qi}^q}{(1+\lambda_i^2)(1+\lambda_p^2)(1+\lambda_q^2)} \\
&= e^{\phi/2} \sum_{i,p,q} \frac{(u_{pi}^q)^2}{(1+\lambda_i^2)(1+\lambda_p^2)(1+\lambda_q^2)} + e^{\phi/2} \sum_{i,p} \frac{1}{(1+\lambda_i^2)(1+\lambda_p^2)} \sum_\alpha (u_{pi}^\alpha)^2 \\
&\quad + 2e^{\phi/2} \sum_{i,p} \frac{\lambda_p^2 (u_{pi}^p)^2}{(1+\lambda_i^2)(1+\lambda_p^2)^2} + e^{\phi/2} \sum_{i,p \neq q} \frac{\lambda_p \lambda_q (u_{pi}^p u_{qi}^q + u_{qi}^p u_{pi}^q)}{(1+\lambda_i^2)(1+\lambda_p^2)(1+\lambda_q^2)}.
\end{aligned}$$

Then the right hand side of the above equation is same to the right hand side of the formula (3.7) in the proposition 3.1 of [13]. Hence, we can use their Theorem 3.1. It says

$$L_g(e^{\phi/2}) \geq K_0 |B|^2 \geq \varepsilon g^{ij} (e^{\phi/2})_i (e^{\phi/2})_j.$$

Here, B is the second fundamental form, and we used $\det(g_{ij}) < 9$ in the above inequality. Now by Lemma 5, we obtain that ϕ is a constant. And the above inequality tells us $B = 0$, which implies M is a linear subspace.

(iii) By the condition and (3.9), we have

$$\begin{aligned}
L_g(\phi) &= -2 \sum_{i,p,q} \frac{\lambda_q^2 (u_{pi}^q)^2}{(1+\lambda_i^2)(1+\lambda_p^2)(1+\lambda_q^2)} + 2 \sum_{\alpha,i,p} \frac{(u_{pi}^\alpha)^2}{(1+\lambda_i^2)(1+\lambda_p^2)} \\
&\quad + 2 \sum_{i,p,q} \frac{\lambda_p \lambda_q u_{pi}^p u_{qi}^q}{(1+\lambda_i^2)(1+\lambda_p^2)(1+\lambda_q^2)} \\
&\geq \sum_i \frac{2}{(1+\lambda_i^2)} \left(\sum_p \frac{\lambda_p u_{pi}^p}{1+\lambda_p^2} \right)^2 = \frac{1}{2} g^{ij} \phi_i \phi_j.
\end{aligned}$$

By Lemma 5, ϕ is a constant. And the above inequality tells us that u^α are linear functions. \square

Corollary 11. *If the normal bundle of the self-shrinking system (1.4) is flat or the codimension of the self-shrinking system (1.4) is 1, then the graph M is a linear subspace.*

Proof. We diagonal u_i^α as (3.8), then $\{e_i\}$ and $\{n_\alpha\}$ are orthonormal. Let $h_{ij}^\alpha = \langle \bar{\nabla}_{e_i} e_j, n_\alpha \rangle$. Then normal bundle flat means

$$h_{qi}^\alpha g^{ij} h_{jp}^\beta = h_{qi}^\beta g^{ij} h_{jp}^\alpha, \quad \text{for any } \alpha, \beta \in \{1, \dots, m\}.$$

Since

$$h_{ij}^\alpha = \langle \bar{\nabla}_{e_i} e_j, n_\alpha \rangle = \langle u_{ij}^\beta E_{n+\beta}, n_\alpha \rangle = \frac{u_{ij}^\alpha}{(1 + |Du^\alpha|^2)^{1/2}},$$

the condition (iii) in Theorem 10 holds, so the graph is a linear subspace. If the codimension is 1, then the condition (iii) in Theorem 10 obviously holds. Therefore we obtain the result. \square

Remark 12. *The second result of above Corollary is firstly proved by L. Wang in [7].*

Now we study the rotational symmetric manifolds corresponding to the origin. This means

$$u^\alpha(x_1, \dots, x_n) = u^\alpha(r),$$

where $r = \sqrt{x_1^2 + \dots + x_n^2}$. Directly calculation shows

$$u_i^\alpha = u_r^\alpha \frac{x_i}{r}, \quad \text{and} \quad u_{ij}^\alpha = u_{rr}^\alpha \frac{x_i x_j}{r^2} + u_r^\alpha \left(\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3} \right).$$

So the metric and the inverse metric become

$$g_{ij} = \delta_{ij} + |u_r|^2 \frac{x_i x_j}{r^2}, \quad \text{and} \quad g^{ij} = \delta_{ij} - \frac{|u_r|^2}{1 + |u_r|^2} \frac{x_i x_j}{r^2},$$

where $|u_r|^2 = \sum_\alpha (u_r^\alpha)^2$. Then, we have

$$g^{ij} u_{ij}^\alpha = \frac{u_{rr}^\alpha}{1 + |u_r|^2} u_{rr}^\alpha + u_r^\alpha \frac{n-1}{r}.$$

So the system (1.4) becomes

$$(3.11) \quad \frac{u_{rr}^\alpha}{1 + |u_r|^2} + u_r^\alpha \frac{n-1}{r} = -u^\alpha + r u_r^\alpha.$$

Proposition 13. *Suppose a smooth solution of the self-shrinking system (1.4) is a rotation symmetry manifold. Then, it should be \mathbb{R}^n plane.*

Proof. We let

$$\phi = \ln(1 + |u_r|^2).$$

For $r \in (0, +\infty)$, using a similar computation of (3.6)-(3.9), we have

$$\begin{aligned} \frac{\phi_{rr}}{1 + |u_r|^2} &= \left(r - \frac{n-1}{r}\right) \phi_r + 2 \frac{n-1}{r^2} \frac{|u_r|^2}{1 + |u_r|^2} + 2 \frac{\sum_\alpha (u_{rr}^\alpha)^2}{1 + |u_r|^2} \\ &\geq \left(r - \frac{n-1}{r}\right) \phi_r + \frac{1}{2} \frac{\phi_r^2}{1 + |u_r|^2}. \end{aligned}$$

Let us assume that in $[0, +\infty)$, ϕ is not a constant. It implies that there is some $R_0 > \sqrt{2n}$, such that in $[0, R_0]$, ϕ is not a constant. By the strong maximum principle, it means that

the maximum point of ϕ only achieves at $r = 0$ or $r = R_0$. For the rotation symmetry manifolds, $|Du|(0) = 0$. In $[0, R_0]$, suppose ϕ achieves its maximum value at $r = 0$. Since $\phi \geq 0$, then $\phi = 0$ in $[0, R_0]$, which is a contradiction. Hence, ϕ achieves its maximum point at $r = R_0$. For $0 < k < 1$, let

$$\eta(r) = \begin{cases} 1 & |r| \leq R_0 \\ -k(r^2 - 2(n-1)\ln r - R_0^2 + (n-1)\ln R_0^2) + 1 & |r| \geq R_0 \end{cases}.$$

By the same argument using in Lemma 5, for sufficiently small k and sufficiently large R_0 , $\eta e^{C\phi}$ only achieves its maximum value in $r > R_0$ and $\eta > 0$. Now at the maximum point,

$$\begin{aligned} e^{-C\phi} \frac{(\eta e^{C\phi})_{rr}}{1 + |u_r|^2} &= \frac{\eta_{rr}}{1 + |u_r|^2} + \frac{2C\eta_r \phi_r}{1 + |u_r|^2} + \frac{C\eta \phi_{rr}}{1 + |u_r|^2} + \frac{C^2 \eta \phi_r^2}{1 + |u_r|^2} \\ &\geq \frac{-2k(1 + \frac{n-1}{r^2})}{1 + |u_r|^2} + C\eta(r - \frac{n-1}{r})\phi_r + C\eta(\frac{1}{2} - C)\frac{\phi_r^2}{1 + |u_r|^2}. \end{aligned}$$

Here, we used $\eta_r + C\eta\phi_r = 0$. Now we choose $C < 1/2$, then

$$e^{-C\phi} \frac{(\eta e^{C\phi})_{rr}}{1 + |u_r|^2} \geq \frac{-2k(1 + \frac{n-1}{r^2})}{1 + |u_r|^2} + 2k(r - \frac{n-1}{r})^2 > 0,$$

which is a contradiction. Then ϕ is a constant which implies $u^\alpha = 0$. \square

4. A SHARP GROWTH ESTIMATE IN EUCLIDEAN SPACE

This section is composed by two parts. In the first part, we give a sharp growth estimate for the system (1.4) in Euclidean space. In the second part, we generalize the previous result. In fact, we will prove a slightly weak result for a class of linear elliptic systems.

Let T be an arbitrary fixed positive constant. For $(x, t) \in \mathbb{R}^n \times [0, T]$, we let (see [17] for codimension 1 case)

$$(4.1) \quad w^\alpha(x, t) = \sqrt{T-t} \, u^\alpha\left(\frac{x}{\sqrt{T-t}}\right) \quad \text{for } \alpha \in \{1, \dots, m\}.$$

In $\mathbb{R}^n \times [0, T]$, let

$$\begin{aligned} (4.2) \quad \bar{g}_{ij}(x, t) &= \delta_{ij} + \sum_{\alpha} \frac{\partial w^\alpha}{\partial x_i} \frac{\partial w^\alpha}{\partial x_j} \\ &= \delta_{ij} + \sum_{\alpha} \frac{\partial u^\alpha}{\partial x_i}\left(\frac{x}{\sqrt{T-t}}\right) \frac{\partial u^\alpha}{\partial x_j}\left(\frac{x}{\sqrt{T-t}}\right) = g_{ij}\left(\frac{x}{\sqrt{T-t}}\right). \end{aligned}$$

A direct calculation shows,

$$\begin{aligned} \frac{\partial w^\alpha}{\partial t} &= -\frac{1}{2\sqrt{T-t}} u^\alpha\left(\frac{x}{\sqrt{T-t}}\right) + x_i \frac{\partial u^\alpha}{\partial x_i}\left(\frac{x}{\sqrt{T-t}}\right) \frac{1}{2(T-t)} \\ &= \frac{1}{2\sqrt{T-t}} \left(-u^\alpha\left(\frac{x}{\sqrt{T-t}}\right) + \frac{x}{\sqrt{T-t}} \cdot Du^\alpha\left(\frac{x}{\sqrt{T-t}}\right) \right) \\ &= \frac{1}{2\sqrt{T-t}} g^{ij}\left(\frac{x}{\sqrt{T-t}}\right) u_{ij}^\alpha\left(\frac{x}{\sqrt{T-t}}\right), \end{aligned}$$

where we used the system (1.4) in the last equality. Define a heat operator,

$$\mathcal{L}_g = \frac{\partial}{\partial t} - \frac{1}{2} \sum_{i,j} \bar{g}^{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

where (\bar{g}^{ij}) is the inverse matrix of (\bar{g}_{ij}) . Note that

$$w_{ij}^\alpha = \frac{1}{\sqrt{T-t}} u_{ij}^\alpha \left(\frac{x}{\sqrt{T-t}} \right).$$

Then we obtain

$$(4.3) \quad \mathcal{L}_g(w^\alpha) = 0.$$

In $\mathbb{R}^n \times [0, T)$, we let

$$\eta(x, t) = 1 - |x|^2 - 3nt, \quad \text{and} \quad |w|^2 = \sum_{\alpha} (w^\alpha)^2.$$

Using (4.3), we have

$$(4.4) \quad \mathcal{L}_g |w|^2 = 2 \sum_{\alpha} w^\alpha \mathcal{L}_g w^\alpha - \sum_{\alpha, i, j} \bar{g}^{ij} w_i^\alpha w_j^\alpha = - \sum_{\alpha, i, j} \bar{g}^{ij} w_i^\alpha w_j^\alpha,$$

and

$$(4.5) \quad \mathcal{L}_g \eta = -3n + \sum_i \bar{g}^{ii}.$$

Now, we are in the position to give the proof of Theorem 2.

Proof of Theorem 2. For any fixed $\rho > 0$, in (4.1), we take

$$T = \frac{1}{12n} + \frac{1}{\rho^2}.$$

In $\mathbb{R}^n \times [0, T)$, we define a function

$$\phi = \eta |w|^2.$$

Assume that ϕ achieves its maximum value at some point $p = (x_0, t_0)$ in the set $\mathbb{R}^n \times [0, \frac{1}{12n}]$, where $\eta(x_0, t_0) > 0$. By the definition of $\eta(x, t)$, we have

$$(4.6) \quad \begin{aligned} \frac{1}{2\rho^2} \sup_{|x| \leq \rho/2} |u(x)|^2 &= \frac{1}{2} \sup_{|x| \leq 1/2} \frac{\sum_{\alpha} |u^\alpha(\rho x)|^2}{\rho^2} \\ &\leq \sup_{|x| \leq 1/2} \left(\sum_{\alpha} \left| \frac{u^\alpha(\rho x)}{\rho} \right|^2 \eta(x, \frac{1}{12n}) \right) \\ &\leq \sup_{x \in \mathbb{R}^n} \phi|_{t=\frac{1}{12n}}. \end{aligned}$$

If $\phi(p) \leq 1$, then

$$(4.7) \quad \sup_x \phi|_{t=\frac{1}{12n}} \leq \phi(p) \leq 1.$$

Combining (4.6), (4.7) and using the arbitrary choice of ρ , we get the theorem.

Since $\phi(p) \geq 1$ and $\eta \leq 1$, we get $|w(p)| \geq 1$. If $t_0 > 0$, then we have

$$(4.8) \quad 2 \sum_{\alpha} w^\alpha \eta D w^\alpha + |w|^2 D \eta = 0$$

at the point p , where D only takes derivatives to space directions as before. Thus we have

$$\begin{aligned}
(4.9) \quad & 0 \leq \mathcal{L}_g(\phi) \\
& = \eta \mathcal{L}_g |w|^2 + |w|^2 \mathcal{L}_g \eta - 2 \sum_{\alpha, i, j} \bar{g}^{ij} w^\alpha w_i^\alpha \eta_j \\
& = -\eta \sum_{\alpha, i, j} \bar{g}^{ij} w_i^\alpha w_j^\alpha + |w|^2 (-3n + \sum_i \bar{g}^{ii}) + 2 \sum_{\alpha, i, j} \bar{g}^{ij} w^\alpha w_i^\alpha \left(\sum_\beta \frac{1}{|w|^2} 2w^\beta \eta w_j^\beta \right) \\
& \leq -\eta \sum_{\alpha, i, j} \bar{g}^{ij} w_i^\alpha w_j^\alpha + |w|^2 (-3n + \sum_i \bar{g}^{ii}) + 4\eta \sum_{\alpha, i, j} \bar{g}^{ij} w_i^\alpha w_j^\alpha \\
& \leq 3\eta (n - \sum_i \bar{g}^{ii}) + (-3n + \sum_i \bar{g}^{ii}) \\
& = -2 \sum_i \bar{g}^{ii} + 3(\eta - 1)(n - \sum_i \bar{g}^{ii}).
\end{aligned}$$

Here we have used (4.4), (4.5) and (4.8) in the third step and the Cauchy inequality in the forth step. In fact, there exists an orthonormal matrix $(P_{ij})_{n \times n}$, such that $\bar{g}_{ij} = P_{ik} \theta_k P_{jk}$, where $\theta_k > 0$ at the point p . Then

$$\begin{aligned}
\sum_{\alpha, \beta, i, j} \bar{g}^{ij} w^\alpha w_i^\alpha w^\beta w_j^\beta & = \sum_{\alpha, \beta, i, j, k} \theta_k w^\alpha w_i^\alpha P_{ik} w^\beta w_j^\beta P_{jk} = \sum_k \theta_k \left(\sum_{\alpha, i} w^\alpha w_i^\alpha P_{ik} \right)^2 \\
& \leq \sum_k \theta_k \left(\sum_\alpha (w^\alpha)^2 \sum_i \left(\sum_i w_i^\alpha P_{ik} \right)^2 \right) = |w|^2 \sum_{\alpha, i, j} \bar{g}^{ij} w_i^\alpha w_j^\alpha.
\end{aligned}$$

Since $0 < \bar{g}^{ii} \leq 1$ and $\eta \leq 1$ by the definition of η , from (4.9), we obtain $\eta(p) \geq 1$ which implies $t_0 = 0$. We have a contradiction. Anyway, ϕ achieves its maximum value at $t = 0$, namely,

$$(4.10) \quad \sup_{x \in \mathbb{R}^n} \phi|_{t=\frac{1}{12n}} \leq \sup_{x \in \mathbb{R}^n} \phi|_{t=0}.$$

Combining (4.6) and (4.10), we get

$$\begin{aligned}
\frac{1}{2\rho^2} \sup_{|x| \leq \rho/2} |u(x)|^2 & \leq \sup_{x \in \mathbb{R}^n} \phi|_{t=\frac{1}{12n}} \\
& \leq \sup_{x \in \mathbb{R}^n} \left((1 - |x|^2) T |u(\frac{x}{\sqrt{T}})|^2 \right).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\sup_{|x| \leq \rho/2} |u(x)|^2 & \leq 2\rho^2 \sup_{|x| \leq 1} T |u(\frac{x}{\sqrt{T}})|^2 \\
& \leq \left(\frac{\rho^2}{6n} + 2 \right) \sup_{|x| \leq 2\sqrt{3n}} |u(x)|^2.
\end{aligned}$$

Using the arbitrary choice of ρ , we obtain the theorem. \square

Remark 14. Assume that u^α is a solution of the system (1.7) satisfying (1.8) and (1.9) for $t = 1$. Let $\bar{a}^{ij}(x, t) = a^{ij}(\frac{x}{\sqrt{T-t}})$, $w^\alpha(x, t) = \sqrt{T-t} u^\alpha(\frac{x}{\sqrt{T-t}})$ and operator

$\mathcal{L}_a = \frac{\partial}{\partial t} - \frac{1}{2} \sum_{i, j} \bar{a}^{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ for any fixed $T > 0$. Then $\mathcal{L}_a w^\alpha = 0$. Let $\eta = 1 - |x|^2 - 3\sigma t$.

Using almost the same proof of the above theorem, we know that $|u|$ is also linear growth.

Corollary 15. *The mean curvature vector is also linear growth. Namely, there exists a positive constant C depending only on n and $\sup_{|x| \leq 2\sqrt{3n}} |u(x)|$, such that*

$$(4.11) \quad |H(x)| \leq C(1 + |x|).$$

Proof. By formula (2.2), we have

$$H^\alpha = \frac{-u^\alpha + x \cdot Du^\alpha}{\sqrt{1 + |Du^\alpha|^2}}.$$

Then using Theorem 2 and the Cauchy inequality, we obtain

$$|H^\alpha(x)| \leq |u^\alpha(x)| + |x| \leq C(1 + |x|),$$

where C is a positive constant depending only on n and $\sup_{|x| \leq 2\sqrt{3n}} |u(x)|$. It implies the estimate. \square

Now let's generalize Theorem 2 to the system (1.7) satisfying (1.8) and (1.9). Firstly, we give some preliminary results. For $s > 0$, we denote

$$(4.12) \quad g(s) = \frac{1}{s+1} \left(\frac{2s}{s+1} \right)^s.$$

Calculating its derivative, we have

$$\frac{g'(s)}{g(s)} = \ln \frac{2s}{s+1} > 0, \quad \text{for } s > 1.$$

And $g(1) = 1/2$, $g(+\infty) = +\infty$. So, there is only one s_0 satisfying

$$g(s_0) = 1.$$

In fact, explicit calculation shows that $3.4 < s_0 < 3.5$.

Lemma 16. *For any $s \geq s_0$, there is some $1 < \zeta < 2$ satisfying*

$$(4.13) \quad \frac{2}{2-\zeta} \leq \zeta^s.$$

Proof. Let

$$f(\zeta) = \zeta^{s+1} - 2\zeta^s + 2.$$

Then the derivative of f is

$$f'(\zeta) = (s+1)\zeta^{s-1} \left(\zeta - \frac{2s}{s+1} \right).$$

We see that

$$\zeta = \frac{2s}{s+1}$$

is a local minimum value of f . Note that

$$f\left(\frac{2s}{s+1}\right) = 2\left[1 - \frac{1}{s+1} \left(\frac{2s}{s+1}\right)^s\right] = 2(1 - g(s)).$$

Since $s \geq s_0$, then $g(s) \geq 1$. We have $f(\zeta) \leq 0$, which implies (4.13). \square

For $s \geq s_0$, denote

$$(4.14) \quad \theta = \sqrt{\frac{s}{s+1}}, \quad k = \sqrt{2}\theta, \quad R_0^2 = \max\left\{r_0, \frac{n\sigma + 1}{2} \frac{k^2}{k^2 - 1}\right\}.$$

Here, r_0 is a radius taking in (1.9). For any fixed $R > R_0$, let

$$(4.15) \quad \bar{R} = R/\theta > R, \quad |u| = \sum_{\alpha} (u^{\alpha})^2.$$

Define two functions

$$(4.16) \quad \eta = \bar{R}^2 - |x|^2, \quad \text{and} \quad \phi = \eta|u|^2.$$

Then ϕ achieves its maximum value in the set $\{x \in \mathbb{R}^n; \eta > 0\}$. We assume the maximum point to be p . At p ,

$$(4.17) \quad \phi_i = \eta_i |u|^2 + 2\eta \sum_{\alpha} u^{\alpha} u_i^{\alpha} = 0.$$

Then

$$a^{ij} \phi_{ij} = a^{ij} \eta_{ij} |u|^2 + 4 \sum_{\alpha} u^{\alpha} a^{ij} \eta_i u_j^{\alpha} + 2\eta \sum_{\alpha} a^{ij} u_i^{\alpha} u_j^{\alpha} + 2\eta \sum_{\alpha} u^{\alpha} a^{ij} u_{ij}^{\alpha}.$$

Using (1.7), (4.16) and (4.17), we get

$$\begin{aligned} a^{ij} \phi_{ij} |u|^2 &= -2a^{ij} \delta_{ij} |u|^4 - 8 \sum_{\alpha, \beta} \eta u^{\alpha} u^{\beta} a^{ij} u_i^{\beta} u_j^{\alpha} + 2\eta |u|^2 \sum_{\alpha} a^{ij} u_i^{\alpha} u_j^{\alpha} \\ &\quad + 2\eta |u|^2 \sum_{\alpha} u^{\alpha} (-u^{\alpha} + x \cdot Du^{\alpha}). \end{aligned}$$

Using the Cauchy inequality,

$$2a^{ij} u^{\alpha} u^{\beta} u_i^{\alpha} u_j^{\beta} \leq a^{ij} u^{\alpha} u^{\alpha} u_i^{\beta} u_j^{\beta} + a^{ij} u_i^{\alpha} u_j^{\alpha} u^{\beta} u^{\beta},$$

then at p , by (4.17), we get

$$\begin{aligned} 0 &\geq -2 \sum_i a^{ii} |u|^4 - 8\eta |u|^2 \sum_{\alpha} a^{ij} u_i^{\alpha} u_j^{\alpha} + 2\eta |u|^2 \sum_{\alpha} a^{ij} u_i^{\alpha} u_j^{\alpha} \\ &\quad + 2\eta |u|^2 [-|u|^2 + \frac{1}{2} x \cdot D|u|^2] \\ &= |u|^2 [-2 \sum_i a^{ii} |u|^2 - 6\eta \sum_{\alpha} a^{ij} u_i^{\alpha} u_j^{\alpha} - 2\eta |u|^2 - x \cdot D\eta |u|^2]. \end{aligned}$$

If at p , $|u| = 0$, then in B_R , $u^{\alpha} = 0$. Hence, in B_R , it is obviously polynomial growth. So we can assume at p , $|u| \neq 0$. Then

$$\begin{aligned} 6\eta \sum_{\alpha} a^{ij} u_i^{\alpha} u_j^{\alpha} &\geq -2 \sum_i a^{ii} |u|^2 - 2\eta |u|^2 - x \cdot D\eta |u|^2 \\ &\geq (4|x|^2 - 2n\sigma - 2\bar{R}^2) |u|^2. \end{aligned}$$

So, we obtain

$$(4|x|^2 - 2n\sigma - 2\bar{R}^2) \phi \leq 6\eta^2 \sum_{\alpha} a^{ij} u_i^{\alpha} u_j^{\alpha} \leq 6c\bar{R}^{2\tau+2}.$$

Hence, we have two cases. The first case is

$$p \in \{x \in \mathbb{R}^n; 2|x|^2 - n\sigma - \bar{R}^2 \geq 1\}.$$

Then

$$\phi(p) \leq 3c\bar{R}^{2\tau+2}.$$

Since ϕ achieves its maximum value at p , we have, in B_R ,

$$3c\bar{R}^{2\tau+2} \geq (\bar{R}^2 - |x|^2) |u|^2 \geq (\bar{R}^2 - R^2) |u|^2.$$

Using (4.15), we obtain

$$(4.18) \quad \sup_{B_R} |u|^2 \leq \frac{3cR^{2\tau}}{\theta^{2\tau}(1-\theta^2)}.$$

The other case is

$$p \in \{x \in \mathbb{R}^n; 2|x|^2 - n\sigma - \bar{R}^2 \leq 1\}.$$

Then, we have

$$(4.19) \quad \sup_{B_R} \phi \leq \sup_{B_r} \phi.$$

Here

$$r = \sqrt{(\bar{R}^2 + n\sigma + 1)/2} = \sqrt{\frac{R^2}{k^2} + \frac{n\sigma + 1}{2}}.$$

By (4.19), we have

$$\left(\frac{R^2}{\theta^2} - R^2\right) \sup_{B_R} |u|^2 \leq \frac{R^2}{\theta^2} \sup_{B_r} |u|^2.$$

Using Lemma 16, we have

$$(4.20) \quad \sup_{B_R} |u|^2 \leq \frac{2}{2-k^2} \sup_{B_r} |u|^2 \leq k^{2s} \sup_{B_r} |u|^2.$$

Combining (4.18) and (4.20), we obtain the following Lemma.

Lemma 17. *Let $s \geq s_0$. For every $R > R_0$, any solution of system (1.7) satisfying (1.8) and (1.9) should satisfy one of the two inequalities: (4.18) or (4.20).*

So, we have the following growth estimate.

Theorem 18. *Assume that u^α is a solution of the system (1.7) satisfying (1.8) and (1.9). Then $|u|$ is polynomial growth. Namely, for $s \geq s_0$, the solution u^α have the estimate,*

$$(4.21) \quad u^\alpha(x) \leq C(1 + \sup_{B_{\sqrt{k^2+1}R_0}} |u|)(1 + |x|^{\max\{s, \tau\}}),$$

where C depending on σ, c, n, s, τ and r_0 .

Proof. Let

$$R = |x| > R_0.$$

There is a nonnegative integer m_0 , such that

$$(4.22) \quad R_0^2 \leq \frac{R^2}{k^{2m_0}} \leq k^2 R_0^2.$$

For $m \geq 2$, denote

$$(4.23) \quad R_m^2 = \frac{R^2}{k^{2(m-1)}} + \frac{n\sigma + 1}{2} \left[1 + \frac{1}{k^2} + \cdots + \frac{1}{k^{2(m-2)}}\right].$$

Let $R_1 = R$. Obviously, for $1 \leq m \leq m_0 + 1$,

$$R_m > R_0.$$

It implies Lemma 17 is applicable for B_{R_m} . Hence, it will appear two cases. The first is that (4.20) holds for every $1 \leq m \leq m_0$. The second is that there is an integer $1 \leq m_1 \leq m_0$,

such that, for any integer $1 \leq m \leq m_1 - 1$, (4.20) holds and (4.18) holds in $B_{R_{m_1}}$. In the first case, for $1 \leq m \leq m_0$, we have

$$(4.24) \quad \sup_{B_{R_m}} |u|^2 \leq k^{2s} \sup_{B_{R_{m+1}}} |u|^2.$$

Iterating (4.24), we get

$$\sup_{B_R} |u|^2 = \sup_{B_{R_1}} |u|^2 \leq k^{2sm_0} \sup_{B_{R_{m_0+1}}} |u|^2.$$

By (4.15), (4.22) and (4.23), we have

$$\begin{aligned} R_{m_0+1}^2 &= \frac{R^2}{k^{2m_0}} + \frac{n\sigma + 1}{2} \left[1 + \frac{1}{k^2} + \cdots + \frac{1}{k^{2(m_0-1)}} \right] \\ &= \frac{R^2}{k^{2m_0}} + \frac{n\sigma + 1}{2} \frac{1 - 1/k^{2m_0}}{1 - 1/k^2} \\ &\leq k^2 R_0^2 + \frac{n\sigma + 1}{2} \frac{1}{1 - 1/k^2} \\ &\leq (k^2 + 1) R_0^2. \end{aligned}$$

Combining the above two inequalities and (4.22), we obtain

$$(4.25) \quad \sup_{B_R} |u|^2 \leq \frac{R^{2s}}{R_0^{2s}} \sup_{B_{\sqrt{k^2+1}R_0}} |u|^2.$$

In the second case, (4.24) holds for $1 \leq m \leq m_1 - 1$, then, similar, we have

$$\sup_{B_R} |u|^2 \leq k^{2s(m_1-1)} \sup_{B_{R_{m_1}}} |u|^2 \leq C k^{2s(m_1-1)} R_{m_1}^{2\tau}.$$

Here, C is a constant depending on c and s . By (4.22), we have

$$k^{2(m_1-1)} \leq \frac{R^2}{R_0^2 k^{2(m_0+1-m_1)}} \leq \frac{R^2}{R_0^2}.$$

Combining the above two inequalities, for $s > \tau$, we get

$$\begin{aligned} \sup_{B_R} |u|^2 &\leq C k^{2(s-\tau)(m_1-1)} \{ k^{2(m_1-1)} \left[\frac{R^2}{k^{2(m_1-1)}} + \frac{n\sigma + 1}{2} (1 + \cdots + \frac{1}{k^{2(m_1-2)}}) \right] \}^\tau \\ &\leq C \frac{R^{2(s-\tau)}}{R_0^{2(s-\tau)}} \left[R^2 + k^{2(m_1-1)} \frac{n\sigma + 1}{2} \frac{1}{1 - 1/k^2} \right]^\tau \\ &\leq C R^{2(s-\tau)} \left[R^2 + \frac{R^2}{R_0^2} R_0^2 \right]^\tau \\ &\leq C R^{2s}. \end{aligned}$$

For $s \leq \tau$, the above second inequality becomes

$$\sup_{B_R} |u|^2 \leq C \left[R^2 + k^{2(m_1-1)} \frac{n\sigma + 1}{2} \frac{1}{1 - 1/k^2} \right]^\tau \leq C R^{2\tau}.$$

Combining the above two inequalities, for $|x| > R_0$, we obtain

$$|u|^2(x) \leq C \left(1 + \sup_{B_{\sqrt{k^2+1}R_0}} |u|^2 \right) |x|^{2\max\{s,\tau\}}.$$

It implies (4.21). □

Remark 19. *The method using in the proof of Theorem 18 also can be used to obtain a growth estimate of $|u|$ for the system (1.4) in Euclidean space. But it is not sharp.*

5. A BERNSTEIN TYPE RESULT OF SPACE-LIKE SELF-SHRINKING GRAPH IN PSEUDO-EUCLIDEAN SPACE

Proposition 20. *In pseudo-Euclidean space with index m , if the eigenvalues of the metric matrix (g_{ij}) have a positive low bound, then M is a linear subspace.*

Proof. In pseudo-Euclidean space, using a similar argument for the volume element function, we obtain

$$\begin{aligned}
 (5.1) \quad & L_g(-\phi) \\
 &= 2 \sum_{\alpha, \beta} g^{ij} g^{mk} g^{nl} u_{mi}^\alpha u_n^\alpha u_{kj}^\beta u_l^\beta - 2 \sum_{\alpha, \beta} g^{ij} g^{mk} g^{nl} u_{ni}^\alpha u_m^\alpha u_{kj}^\beta u_l^\beta \\
 &\quad + 2 \sum_{\alpha} g^{ij} g^{mn} u_{mi}^\alpha u_{nj}^\alpha \\
 &= \sum_{\alpha, \beta} g^{ij} g^{mk} g^{nl} (u_{mi}^\alpha u_n^\alpha - u_{ni}^\alpha u_m^\alpha) (u_{kj}^\beta u_l^\beta - u_{lj}^\beta u_k^\beta) + 2 \sum_{\alpha} g^{ij} g^{mn} u_{mi}^\alpha u_{nj}^\alpha \\
 &\geq 2 \sum_{\alpha} g^{ij} g^{mn} u_{mi}^\alpha u_{nj}^\alpha \\
 &\geq \varepsilon g^{ij} (-\phi)_i (-\phi)_j.
 \end{aligned}$$

By Lemma 5, we get that $-\phi$ is a constant. Then the first inequality of (5.1) implies $u_{ij}^\alpha = 0$. This is the result. \square

Now we continue to consider spacelike self-shrinker graph M described by (1.3) in pseudo-Euclidean space with index m . In what following, we denote the pseudo-inner product $\langle \cdot, \cdot \rangle$ by

$$\langle X, X \rangle = \sum_i x_i^2 - \sum_{\alpha} (y^\alpha)^2, \quad \text{for each } X = (x_1, \dots, x_n; y^1, \dots, y^m).$$

We write $|X|^2 = \langle X, X \rangle$ for $X \in \mathbb{R}^{n+m}$. We assume that the metric of M is the induced metric.

For any $p \in \mathbb{R}^n$, we choose a coordinate system as (3.8). Then, there are orthonormal vectors $\{e_i(p)\}_{i=1}^n \subset T_p M$, $\{n_\alpha(p)\}_{\alpha=1}^m \subset N_p M$ defined as

$$(5.2) \quad e_i(p) = \frac{1}{\sqrt{1 - \lambda_i^2}} (E_i + \lambda_i E_{n+i}), \quad n_\alpha(p) = \frac{1}{\sqrt{1 - \lambda_\alpha^2}} (\lambda_\alpha E_\alpha + E_{n+\alpha}),$$

where spacelike implies $|\lambda_i| < 1$ for every i . Then we can choose a normal frame $\{e_i\}_{i=1}^n \subset \Gamma(TM)$ and an orthonormal frame $\{n_\alpha\}_{\alpha=1}^m \subset \Gamma(NM)$ locally, such that $e_i(p)$ and $n_\alpha(p)$ is defined by (5.2). We will use these notations in the following, which are different from the definitions in section 2.

The second fundamental form and mean curvature vector are,

$$B_{ij} = \bar{\nabla}_{e_i}^N e_j = \sum_{\alpha} h_{ij}^\alpha n_\alpha, \quad \text{and} \quad H = \sum_i \bar{\nabla}_{e_i}^N e_i.$$

Denote the standard n -form in \mathbb{R}^n by

$$dx = dx_1 \wedge \dots \wedge dx_n.$$

Then, we define a function,

$$*dx = dx(e_1, \dots, e_n).$$

Then, at the point p , we have

$$(5.3) \quad \langle \bar{\nabla}_{e_i} \bar{\nabla}_{e_i} e_j, e_j \rangle = \bar{\nabla}_{e_i} \langle \bar{\nabla}_{e_i} e_j, e_j \rangle - \langle \bar{\nabla}_{e_i} e_j, \bar{\nabla}_{e_i} e_j \rangle = \sum_{\alpha} (h_{ij}^{\alpha})^2.$$

Since $\bar{\nabla}_{e_i} e_j - \bar{\nabla}_{e_j} e_i = [e_i, e_j] \in TM$, then we have, at p ,

$$\begin{aligned} & \langle \bar{\nabla}_{e_i} \bar{\nabla}_{e_i} e_j, n_{\alpha} \rangle - \langle \bar{\nabla}_{e_i} \bar{\nabla}_{e_j} e_i, n_{\alpha} \rangle \\ &= \langle \bar{\nabla}_{e_i} [e_i, e_j], n_{\alpha} \rangle = -\langle [e_i, e_j], \bar{\nabla}_{e_i} n_{\alpha} \rangle = 0. \end{aligned}$$

Since \mathbb{R}^{n+m} is flat, using (1.2) and the above equality, we have, at p ,

$$\begin{aligned} (5.4) \quad & \sum_i \langle \bar{\nabla}_{e_i} \bar{\nabla}_{e_i} e_j, n_{\alpha} \rangle \\ &= \sum_i \langle \bar{\nabla}_{e_i} \bar{\nabla}_{e_j} e_i, n_{\alpha} \rangle = \sum_i \langle \bar{\nabla}_{e_j} \bar{\nabla}_{e_i} e_i, n_{\alpha} \rangle = \langle \bar{\nabla}_{e_j} H, n_{\alpha} \rangle \\ &= -\left\langle \bar{\nabla}_{e_j} \left(X - \sum_k \langle X, e_k \rangle e_k \right), n_{\alpha} \right\rangle \\ &= \sum_k \langle X, e_k \rangle \langle \bar{\nabla}_{e_j} e_k, n_{\alpha} \rangle = -\sum_k \langle X, e_k \rangle h_{jk}^{\alpha}. \end{aligned}$$

By the definition of dx and (5.2), we have, at p ,

$$\begin{aligned} (5.5) \quad & \nabla_{e_i} * dx = \sum_j dx(e_1, \dots, \bar{\nabla}_{e_i} e_j, \dots, e_n) \\ &= \sum_j h_{ij}^{\alpha} dx(e_1, \dots, \underbrace{n_{\alpha}}_j, \dots, e_n) = \sum_j h_{ij}^j \lambda_j * dx. \end{aligned}$$

Let us calculate the Laplace of $*dx$, which is similar to the minimal submanifolds case (see [19]). Combining (5.2)-(5.4), we obtain

$$\begin{aligned} (5.6) \quad & \Delta * dx \\ &= \sum_{i,j \neq k} dx(e_1, \dots, \bar{\nabla}_{e_i} e_j, \dots, \bar{\nabla}_{e_i} e_k, \dots, e_n) + \sum_{i,j} dx(e_1, \dots, \bar{\nabla}_{e_i} \bar{\nabla}_{e_i} e_j, \dots, e_n) \\ &= \sum_{i,j \neq k} h_{ij}^{\alpha} h_{ik}^{\beta} dx(e_1, \dots, \underbrace{n_{\alpha}}_j, \dots, \underbrace{n_{\beta}}_k, \dots, e_n) + \sum_{i,j} \langle \bar{\nabla}_{e_i} \bar{\nabla}_{e_i} e_j, e_j \rangle dx(e_1, \dots, e_n) \\ &\quad - \sum_{i,j} \langle \bar{\nabla}_{e_i} \bar{\nabla}_{e_i} e_j, n_{\alpha} \rangle dx(e_1, \dots, \underbrace{n_{\alpha}}_j, \dots, e_n) \\ &= \sum_{i,j \neq k} \lambda_j \lambda_k h_{ij}^j h_{ik}^k * dx - \sum_{i,j \neq k} \lambda_j \lambda_k h_{ij}^k h_{ik}^j * dx + \sum_{i,j,\alpha} (h_{ij}^{\alpha})^2 * dx \\ &\quad + \sum_{j,k} \langle X, e_k \rangle h_{jk}^j \lambda_j * dx. \end{aligned}$$

We define a second order differential operator (see [5] for Euclidean space),

$$P = \Delta - \langle X, \nabla \cdot \rangle.$$

Using the Cauchy inequality, since $|\lambda_i| < 1$, we have

$$(5.7) \quad \left| \sum_{i,j \neq k} \lambda_j \lambda_k h_{ij}^k h_{ik}^j \right| \leq \frac{1}{2} \sum_{i,j \neq k} ((h_{ij}^k)^2 + (h_{ik}^j)^2) = \sum_{i,j \neq k} (h_{ij}^k)^2.$$

Then (5.6) becomes

$$(5.8) \quad \begin{aligned} P(*dx) &= \left(\sum_{i,j \neq k} \lambda_j \lambda_k h_{ij}^j h_{ik}^k - \sum_{i,j \neq k} \lambda_j \lambda_k h_{ij}^k h_{ik}^j + \sum_{i,j,\alpha} (h_{ij}^\alpha)^2 \right) * dx \\ &\geq \left(\sum_{i,j \neq k} \lambda_j \lambda_k h_{ij}^j h_{ik}^k + \sum_{i,k} \lambda_i^2 (h_{ik}^i)^2 \right) * dx \\ &= \left(\sum_{i,j,k} \lambda_j \lambda_k h_{ij}^j h_{ik}^k \right) * dx = \frac{|\nabla * dx|^2}{*dx}. \end{aligned}$$

We are in the position to give the proof of Theorem 3.

Proof of Theorem 3. Denote the induced metric in M by $g = \sum_{i,j} g_{ij} dx_i dx_j$, seeing (2.3), and $\det g = \det(g_{ij})$. Denote weighted function ρ by

$$\rho = \exp\left(-\frac{|X|^2}{2}\right) = \exp\left(-\frac{|x|^2 - |u(x)|^2}{2}\right),$$

and the volume element of M by

$$d\mu = \sqrt{\det g} dx_1 \wedge \cdots \wedge dx_n = \sqrt{\det g} dx.$$

Then, for any local orthonormal frame $\{e_i\}_{i=1}^n$ of tangent bundle TM , we have

$$*dx = \frac{1}{\sqrt{\det g}}, \quad \text{and} \quad d\mu(e_1, \dots, e_n) = 1.$$

Since $2\langle X, \nabla \cdot \rangle = \langle \nabla |X|^2, \nabla \cdot \rangle$, then the operator P is invariant under orthonormal transformations from \mathbb{R}^n to \mathbb{R}^n and from \mathbb{R}^m to \mathbb{R}^m . Hence, (5.8) holds in the whole M . Let $f = \frac{1}{\sqrt{\det g}}$ and $\eta \in C_c^\infty(M)$. We will determine η later. By (5.8), we have

$$(5.9) \quad \begin{aligned} \int_M \frac{|\nabla f|^2}{f} \eta^2 \rho d\mu &\leq \int_M P(f) \eta^2 \rho d\mu = \int_M \operatorname{div}(\rho \nabla f) \eta^2 d\mu \\ &= 2 \int_M \eta (\nabla f \cdot \nabla \eta) \rho d\mu \leq \frac{1}{2} \int_M \frac{|\nabla f|^2}{f} \eta^2 \rho d\mu + 2 \int_M |\nabla \eta|^2 f \rho d\mu. \end{aligned}$$

Here, 'div' is the divergence of M . Then

$$(5.10) \quad \int_M \frac{|\nabla f|^2}{f} \eta^2 \rho d\mu \leq 4 \int_{\mathbb{R}^n} |\nabla \eta|^2 \rho dx = 4 \int_{\mathbb{R}^n} g^{ij} \eta_i \eta_j \exp\left(-\frac{|x|^2 - |u(x)|^2}{2}\right) dx.$$

On the other hand, there exists a constant κ satisfying $0 < \kappa < 1$, such that $|\lambda_i| \leq \kappa$ for any $x \in \overline{B_1(0)} \subset \mathbb{R}^n, i \in \{1, \dots, n\}$. If $|u(x)| \neq 0$, using $u(0) = 0$, then there exists a nonnegative number $a < 1$, such that, $u(sx) \neq 0$ for any $s \in (a, 1]$ and $u(ax) = 0$. Hence,

for $p \in [ax, x]$,

$$\begin{aligned}
 (5.11) \quad \left| \sum_i x_i \frac{\partial}{\partial x_i} (\sum_{\alpha} (u^{\alpha})^2) \right| &= 2 \left| \sum_{i,\alpha} x_i u^{\alpha} u_i^{\alpha} \right| \\
 &\leq 2 \left(\sum_{\alpha} (u^{\alpha})^2 \right)^{1/2} \left(\sum_{\alpha,i} x_i^2 (u_i^{\alpha})^2 \right)^{1/2} \\
 &\leq 2\kappa |x| |u|.
 \end{aligned}$$

Here we use the fact that every diagonal entry of a positive matrix is no large than the maximum eigenvalue of this matrix. For any $x \in \overline{B_1(0)}$, using the above inequality, we get

$$\begin{aligned}
 (5.12) \quad |u(x)| &= \int_a^1 \frac{\partial}{\partial t} |u(tx)| dt = \int_a^1 \frac{1}{2|u(tx)|} \frac{\partial}{\partial t} |u(tx)|^2 dt \\
 &= \int_a^1 \sum_i \frac{x_i}{2|u(tx)|} \frac{\partial}{\partial x_i} (\sum_{\alpha} (u^{\alpha})^2) |_{tx} dt \leq \int_a^1 \kappa |x| dt \leq \kappa.
 \end{aligned}$$

For any $x \in \mathbb{R}^n \setminus B_1(0)$, $u(x) \neq 0$, there is some $b \geq 1/|x|$, such that, for any $\tilde{b} \geq 1/|x|$, and $u(sx) \neq 0$ holding for any $s \in (\tilde{b}, 1]$, we have $b \leq \tilde{b}$. Then, by (5.11), we have

$$\begin{aligned}
 (5.13) \quad |u(x)| &\leq \left| \int_b^1 \frac{d}{dt} |u(tx)| dt \right| + \left| u\left(\frac{x}{|x|}\right) \right| \leq \int_b^1 \frac{x_i}{2|u(tx)|} \frac{\partial}{\partial x_i} (\sum_{\alpha} (u^{\alpha})^2) |_{tx} dt + \kappa \\
 &\leq \int_b^1 |x| dt + \kappa \leq |x|(1-b) + \kappa \leq |x| - 1 + \kappa.
 \end{aligned}$$

Combining (5.12) and (5.13), we get

$$(5.14) \quad |u(x)|^2 \leq |x|^2 - 2(1-\kappa)|x| + 2, \quad \text{for all } x \in \mathbb{R}^n.$$

Combining (5.10) and (5.14), we obtain

$$(5.15) \quad \int_M \frac{|\nabla f|^2}{f} \eta^2 \rho \leq 4 \int_{\mathbb{R}^n} g^{ij} \eta_i \eta_j e^{1-(1-\kappa)|x|} dx \leq 4 \int_{\mathbb{R}^n} \frac{|\nabla \eta|^2}{\det(g)} e^{1-(1-\kappa)|x|} dx.$$

Now we choose η . For any positive r , let $\eta = \eta(|x|)$ be a cut-off function in \mathbb{R} , $\eta \equiv 1$ in $[0, r]$, $\eta \equiv 0$ in $[2r, +\infty)$, and $|\eta'| \leq C/r$, where C is some constant not depending on r . Since $\det(g)$ has subponential decay in $|x|$ (c.f. Theorem 3), letting r go to infinity in (5.15), we have $\nabla f = 0$. Then $*dx$ is a constant. By (5.7) and (5.8), we obtain $h_{ij}^{\alpha} = 0$, which implies the theorem. \square

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